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Non-axisymmetric instability of a rotating layer of fluid

By Y. T. FUNG
Fluid Dynamics Branch, Marine Technology Division,
Naval Research Laboratory, Washington, D.C. 20375

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A mechanism previously proposed for the possible interaction between a rotating sheet of fluid and a rotating environment is re-investigated using the normal-mode method. Only azimuthal modes are considered. Stability of such a flow, if any, will appear as the neutral condition, i.e. the stability domain will be present as a stability boundary. The stability boundary for the present flow exists only for one azimuthal mode with no density inhomogeneity. For general heterogeneous rotating flows subject to non-axisymmetric disturbances, a semi-ellipse theorem is derived with a restriction. In contrast with the semicircle theorem in two-dimensional stratified flows, the restriction suggests that the semicircle in the complex phase-velocity plane does not in general provide an upper bound on all unstable waves. For rigidly rotating flows all unstable waves lie on a semicircle in the complex phase-velocity plane regardless of the density distributions.

1. Introduction

In a paper on the non-axisymmetric instability of a rotating sheet of fluid in a rotating environment, Sakurai (1976; henceforth referred to as I) presented a mechanism for possible interaction between the sheet and the environment. According to his arguments, when the sheet moved outwards to a new radial location it was squeezed by a higher pressure at its new position and its thickness was decreased. This decrease of thickness led to an outward resultant pressure and, as a result, the sheet was moved further outwards. It was also conjectured that inward movement of the sheet was augmented in a similar way. To understand such an interaction better, he also treated the problem analytically by using the normal-mode approach. The stability domains, in which he classified the unstable waves as stationary and travelling, were obtained. In addition, the transfer of angular momentum between the sheet and the environment, which affected the location of stability boundaries, was also discussed.

The interaction mechanism proposed in I is plausible if immiscibility exists between the sheet and the environment. The effect of surface tension between two immiscible fluids always stabilizes non-axisymmetric perturbations. Perturbations to the sheet will therefore cause it to be squeezed or expanded at a new location. Also because of immiscibility, such squeezing or expanding of the deformed sheet forces fluid out of or into that part of the sheet instead of mixing with the environment.

For miscible fluids such as the flow considered in I, however, such an interaction mechanism is rather implausible because of the absence of immiscibility. Any deformation of the sheet may cause interchange of particles between the sheet and the environment and upset the steady-state profile. In that particular profile for the rotating sheet, two discontinuity interfaces are present in the flow. Instabilities of the Rayleigh–Taylor type or of the Kelvin–Helmholtz type or of both are very likely to occur at either one of the interfaces because of the sharp density and velocity gradients conveyed by such discontinuities. This is exactly the instability mechanism in I rather than the squeeze mechanism proposed in that paper.

In performing the stability analysis in I, the unstable waves were classified as stationary and travelling with respect to the rotating coordinate system. The stability boundaries for each type of wave were discussed separately. We have found such stability boundaries to be incorrect.

For the rotating sheet under consideration, discontinuities exist at two radial locations and constant properties are assumed within the sheet and within the environment. The sheet can therefore be imagined as a 'mirror' to reflect the stability influence on both sides of the sheet, i.e. stabilizing effects imposed on one side of the sheet imply destabilizing effects on the other side. For simplicity, let us first assume that the flow is neutrally stable for some density and velocity ratios between the sheet and the environment. Stability curves for those ratios, of course, coincide with stability boundaries. Now we introduce some stabilizing effects at one interface by changing the velocity and density ratios. The flow is expected to be stabilized at that interface, and the corresponding stability curves will move away from the stability boundary into the stability domain. On the other hand, the same stabilizing effects on one side of the sheet imply destabilizing effects on the other side of the sheet. The stability curves for the other interface would also move away from the stability boundary, but into the instability domain instead. Based on this argument, the result of such a change of velocity and density ratios will always lead to instability. Stability of the flow, if any, must lie on the stability boundary. In other words, there will be no domain for stability except for that which corresponds to stability boundaries.

We will re-examine the profile first by investigating the instability characteristics of the governing stability equation. It will be shown from the semi-ellipse theorem to be derived that stationary unstable waves are impossible if no counterflows exist within the flow domain. This characteristic, as also supported by re-examining the special cases of the secular relation for stability of the rotating sheet, serves as a counter to the stability/instability domains obtained in I. It will also be seen that the stability for those cases corresponds only to the neutral conditions as previously discussed. The semi-ellipse theorem is proved to be valid with a restriction. In contrast with the case of two-dimensional stratified flows, the semicircle theorem for rotating flows in general does not provide a bound for all unstable waves. This characteristic may be seen from the derivation of the theorem and is also supported by an exact solution to the governing stability equation. For rigidly rotating flows, all unstable waves lie on a semicircle in the complex phase-velocity plane regardless of the density distributions.

2. Bounds on unstable waves

Let the axis of a cylindrical coordinate coincide with the axis of symmetry in the basic state of a rotating flow with angular velocity $\Omega(r)$ and density $\rho_0(r)$. The governing equation for stability of the flow subject to azimuthal perturbations is

$$D(\rho_0 r^2 D^* u) + \left\{ -\rho_0 m^2 - \frac{rD[\rho_0 D^*(r\Omega)]}{\Omega - \omega/m} + \frac{r\Omega^2 D\rho_0}{(\Omega-\omega/m)^2} \right\} u = 0, \tag{1}$$

where u is the velocity perturbation in the radial direction; ω is the complex frequency; m is the azimuthal wavenumber, a positive integer; $D = d/dr$ and $D^* = D + 1/r$. We refer to Fung & Kurzweg (1975) for the assumptions and derivation of the equation. The boundary conditions for (1) are $u(r_1) = u(r_2) = 0$, where r_1 and r_2 are the radial positions of rigid boundaries.

We are to obtain bounds on possible unstable waves using the classical integral method. We will also point out, for certain velocity and density profiles, that the bounds are the best possible. Let

$$u = (\Omega - c)\,\psi, \tag{2}$$

where

$$c = c_r + ic_i = \frac{\omega}{m}$$

is the angular phase-velocity. Substituting (2) into (1), multiplying the resultant equation by $r\bar{\psi}$, where $\bar{\psi}$ is the complex conjugate of ψ, and integrating the final equation over the flow region, one obtains

$$\int \rho_0(\Omega - c)^2\, Q\, dr + \int [2rD(\rho_0\Omega)(\Omega - c) - r\Omega^2(D\rho_0)]\,|\psi|^2\, r\, dr = 0, \tag{3}$$

where

$$Q = [r^2 |D^*\psi|^2 + m^2 |\psi|^2]\, r \geqslant 0.$$

We further let

$$Q_1 = \rho_0[r^2 |D\psi|^2 + (m^2 - 1)|\psi|^2]\, r \geqslant 0. \tag{4}$$

Equation (3) than can be reduced to the relatively simple form

$$\int (\Omega^2 - 2c\Omega)\, Q_1\, dr + c^2 \int Q\, dr = 0, \tag{5}$$

with the real and imaginary parts equal to

$$\int \Omega^2 Q_1\, dr - 2c_r \int \Omega Q_1\, dr + (c_r^2 - c_i^2) \int Q\, dr = 0, \tag{6}$$

$$\int \Omega Q_1\, dr - c_r \int Q\, dr = 0. \tag{7}$$

Let $a \leqslant \Omega \leqslant b$, where a and b are respectively the lower and upper bounds of the angular velocity, so that

$$\int (\Omega - a)(\Omega - b)\, Q_1\, dr \leqslant 0. \tag{8}$$

Incorporating (4), (6) and (7) into (8), one obtains

$$\{c_r^2 + c_i^2 - (a+b)\, c_r + ab\} \int Q\, dr + ab \int (D\rho_0)\, r^2 |\psi|^2\, dr \leqslant 0. \tag{9}$$

Equation (9) suggests that the complex angular phase-velocity will no longer be bounded by a semicircle if $ab(D\rho_0) < 0$. This characteristic of the system will be demonstrated by an analytical solution to the stability equation to be given in §3. Meanwhile, we will proceed to construct a semi-ellipse theorem for rotating flows. Let

$$u = (\Omega - c)^{\frac{1}{2}}\,\phi. \tag{10}$$

Substituting (10) into (1) and multiplying the resultant equation by $r\bar{\phi}$, where $\bar{\phi}$ is the complex conjugate of ϕ, we obtain, from the imaginary part of the integral equation,

$$\int \rho_0[r^2 |D^*\phi|^2 + m^2 |\phi|^2]\, r\, dr + \int (J - \tfrac{1}{4}) \frac{\rho_0\, r^2 (D\Omega)^2 |\phi|^2}{|\Omega - c|^2}\, r\, dr = 0, \tag{11}$$

where

$$J = \frac{r\Omega^2(D\rho_0)}{\rho_0 r^2(D\Omega)^2}$$

is the analogue of the local Richardson number encountered in two-dimensional stratified flows. Following the approach used by Kochar & Jain (1979), we obtain from (2), (10) and (11)

$$[1+(1-4J_{\text{min}})^{\frac{1}{2}}]^2 \int r^2(D\Omega)^2\rho_0\,|\psi|^2\, r\,dr \geqslant 4c_i^2 \int Q\,dr, \tag{12}$$

where J_{min} is the minimum value of J. Assuming $ab(D\rho_0) \geqslant 0$ and recalling that $\Omega_{\text{max}} = b$, we combine (9) and (12) to obtain

$$\left\{c_r - \frac{a+b}{2}\right\}^2 + \left\{1 + \frac{a}{b}\frac{4J_{\text{min}}}{[1+(1-4J_{\text{min}})^{\frac{1}{2}}]^2}\right\} c_i^2 \leqslant \frac{(b-a)^2}{4}. \tag{13}$$

Thus the complex angular phase-velocity for unstable waves is bounded by a semi-ellipse described by (13) provided that $ab(D\rho_0) \geqslant 0$, i.e. $abJ \geqslant 0$. In contrast with the semi-ellipse bound in two-dimensional stratified flows, the minor axis of the semi-ellipse bound for non-axisymmetric instabilities of rotating flows depends not only on the distribution of density but also on the upper and lower bounds of the azimuthal velocity.

The above semi-ellipse bound is valid as long as $J < \frac{1}{4}$. However, for uniformly rotating flows $J \rightarrow \infty$, the range of validity excludes the prediction of unstable waves for some important flow phenomena such as those in gaseous centrifuges or other rotational systems. Special consideration will have to be taken in order to obtain a bound for those unstable waves.

For uniformly rotating flows, $\Omega(r) = \Omega_0$, where Ω_0 is a constant. Letting $u = \Psi/r$ transforms (1) to

$$D(\rho_0 rD\Psi) - \left[\frac{\rho_0 m^2}{r} + \Lambda(D\rho_0)\right]\Psi = 0, \tag{14}$$

where

$$\Lambda = \Lambda_r + i\Lambda_i = \frac{2\Omega_0}{\Omega_0 - c} - \frac{\Omega_0^2}{(\Omega_0 - c)^2}, \tag{15}$$

$$\Lambda_r = \frac{\Omega_0}{|\Omega_0 - c|^4}[(\Omega_0 - c_r)^2\,(\Omega_0 - 2c_r) + c_i^2(3\Omega_0 - 2c_r)],$$

$$\Lambda_i = \frac{2\Omega_0\,c_i}{|\Omega_0 - c|^4}[(c_r - \tfrac{1}{2}\Omega_0)^2 + c_i^2 - (\tfrac{1}{2}\Omega_0)^2].$$

Equation (14) plus the boundary conditions forms a Sturm–Liouville system having the following two characteristics: (i) Λ is always real, i.e. $\Lambda_i = 0$, and (ii) Λ and $D\rho_0$ are of opposite signs. These characteristics can also be obtained by the integral method. Rewrite (15) as

$$c = \Omega_0\left[1 - \frac{1}{1 \pm (1-\Lambda)^{\frac{1}{2}}}\right]. \tag{16}$$

The second characteristic and (16) clearly demonstrate that instabilities are impossible when $D\rho_0 \geqslant 0$. The first characteristic shows that the complex phase-velocity for unstable waves, which correspond to $c_i > 0$ and $D\rho_0 < 0$, must lie on a semicircle described by

$$(c_r - \tfrac{1}{2}\Omega_0)^2 + c_i^2 = (\tfrac{1}{2}\Omega_0)^2. \tag{17}$$

Instabilities of this type are certainly of centrifugal origin since no shear layers exist in the flow.

In view of the bounds on unstable waves provided by (13) and (17), the domains of instability in I appear to be incorrect. No unstable disturbances may be expected to be stationary with respect to the non-rotating coordinate system used in this paper or with respect to the rotating coordinate system used in I, if the fluid rotates unidirectionally. In addition, when the sheet and the environment had the same angular velocity, the flow considered in I was stable for the $m = 1$ mode regardless of the density difference between the sheet and the environment. Such a result is rather implausible. As pointed out in the previous discussion on uniformly rotating motion, instabilities will be expected if negative density gradients appear in the flow. As a matter of fact, it will be demonstrated by an exact solution to the stability, equation that the flow profile considered in I is always unstable for all azimuthal modes except when the density of the sheet is the same as that of the environment – this is opposite to the findings in I.

We will proceed to construct a flow profile to demonstrate the validity of the bounds described by (13) and (17), and to re-examine the instability domain for a rotating sheet in a rotating environment.

3. Three rotating layers of fluids

Consider a three-region flow having the following velocity and density distributions:

$$\Omega(r) = \begin{cases} \Omega_1, \\ \Omega_2, \\ \Omega_3, \end{cases} \quad \rho_0(r) = \begin{cases} \rho_1 & (0 \leqslant r < R_1), \\ \rho_2 & (R_1 \leqslant r < R_2), \\ \rho_3 & (R_2 \leqslant r < \infty). \end{cases}$$

Here Ω_k and ρ_k ($k = 1, 2, 3$) are constants. The secular relation obtained by matching both the kinematic and dynamic interfacial conditions (including the unbalanced centrifugal forces arising from the discontinuities of the velocity and density at the interface) is found to be

$$\frac{n_1(n_1-2)-\gamma_{21}n_2(n_2-2)+m(1-\gamma_{21})}{n_1(n_1-2)+\gamma_{21}n_2(n_2+2)+m(1-\gamma_{21})}\left(\frac{R_1}{R_2}\right)^{2m} - \frac{n_2(n_2-2)+\gamma_{32}n_3(n_3+2)+m(1-\gamma_{32})}{n_2(n_2+2)-\gamma_{32}n_3(n_3+2)-m(1-\gamma_{32})} = 0, \quad (18)$$

where

$$n_k = m - \frac{\omega}{\Omega_k} = \frac{m}{\Omega_k}(\Omega_k - c),$$

$$\gamma_{jk} = \frac{\rho_j \Omega_j^2}{\rho_k \Omega_k^2} \quad \text{(no summation)} \quad (j,\ k = 1,\ 2,\ 3,).$$

We will examine the secular relation (18) for three special cases.

3.1. *Uniform rotation*

The first special case to be examined is the flow in which all three layers of fluids rotate at the same speed. The secular relation (18) for $\Omega_k = \Omega_0$ ($k = 1, 2, 3$) reduces to

$$\Delta\left(1-\frac{\rho_2}{\rho_1}\right)\left(1-\frac{\rho_3}{\rho_2}\right)\Lambda^2 - m(1+\Delta)\left(1-\frac{\rho_3}{\rho_1}\right)\Lambda + m^2\left[\left(1+\frac{\rho_3}{\rho_1}\right)+\Delta\left(\frac{\rho_2}{\rho_1}+\frac{\rho_3}{\rho_2}\right)\right] = 0, \quad (19)$$

where

$$\Delta = \frac{R_2^{2m}-R_1^{2m}}{R_2^{2m}+R_1^{2m}} \quad (1 > \Delta \geqslant 0).$$

From (16) and (19), instabilities occur when

$$\frac{m(1+\Delta)\left(1-\frac{\rho_3}{\rho_1}\right) \pm \left\{m^2(1+\Delta)^2\left(1-\frac{\rho_3}{\rho_1}\right)^2-4\Delta m^2\left(1-\frac{\rho_2}{\rho_1}\right)\left(1-\frac{\rho_3}{\rho_2}\right)\left[\left(1+\frac{\rho_3}{\rho_1}\right)+\Delta\left(\frac{\rho_2}{\rho_1}+\frac{\rho_3}{\rho_2}\right)\right]\right\}^{\frac{1}{2}}}{2\Delta\left(1-\frac{\rho_2}{\rho_1}\right)\left(1-\frac{\rho_3}{\rho_2}\right)} > 1. \tag{20}$$

Four types of density distributions exist for the flow under consideration. They are: (*a*) $\rho_3 > \rho_2 > \rho_1$; (*b*) $\rho_3 < \rho_2 < \rho_1$; (*c*) $\rho_3 < \rho_2 > \rho_1$; (*d*) $\rho_3 > \rho_2 < \rho_1$. It can be shown, except for type (*a*), that inequality (20) can always be satisfied and the flow is unstable for all modes regardless of the thickness of the middle layer. Instabilities of these types are certainly of centrifugal origin. The unbalanced centrifugal force due to the decrease of density in either the inner or the outer interface will induce instability as predicted. Furthermore, it can also be shown that the complex phase speeds for all these three types of density distributions lie on a semicircle given by (17). For the density distribution of type (*a*), inequality (20) can never be satisfied, and the flow is certainly stable since the density increases radially outward.

It should be mentioned that the density profiles used in I belong to types (*c*) and (*d*). According to the stability domains discussed in that paper, the flow should be stable at least for the $m = 1$ mode when the sheet and the environment rotate at the same speed. Such a conclusion is certainly incorrect. For density distributions as those in types (*c*) and (*d*) negative density gradients always exist in either the inner or the outer interface. As demonstrated by the instability inequality (20) and by the instability criteria previously discussed, the flow is unstable for all modes despite the thickness of the middle layer except when $\rho_1 = \rho_2 = \rho_3$.

3.2. *The rotating sheet*

The second special case to be investigated is the flow profile in I, a homogeneous rotating sheet of fluid in a rotating environment with different density. The secular relation (18) for $\rho_1 = \rho_3$ and $\Omega_1 = \Omega_3$ reduces to

$$(n_1^2-\alpha\beta^2 n_2^2)^2+2\alpha\beta^2(1+1/\Delta)n_1^2 n_2^2-[2(n_1-\alpha\beta^2 n_2)-m(1-\alpha\beta^2)]^2 = 0, \tag{21}$$

where $\alpha = \rho_2/\rho_1$ and $\beta = \Omega_2/\Omega_1$. Since Sakurai used a coordinate system rotating with the environment, and a timescale normalized with the rotating speed, the complex frequency ω in this paper is then related to the complex frequency σ in I by

$$\omega = (m-\sigma)\Omega_1; \tag{22}$$

the dispersion relation for stability governed by equation (21) of I can easily be obtained by substituting (22) into our equation (21). It is difficult in general to obtain explicit solutions to (21). Assuming the thickness of the sheet to be small and classifying the solutions into travelling and stationary disturbances as the approach used in I seemed to be a way to attack the problem. However, such an assumption and a classification are very questionable because of the dimensional dependence of the complex phase velocity on the sheet thickness. This behaviour can be seen from two exact solutions to be given below.

Before obtaining explicit solutions to (21), we would like to point out that two special cases, which happen to be the least-unstable ones, are of particular interest. They are the case in which the sheet and the environment rotate at the same speed and the case in which the sheet and the environment have the same density. In the former case, shear effects, which always upset flow stability, are absent. In the latter case, centrifugally unbalanced forces that are induced by negative density gradients at either the inner or the outer interface are eliminated. Stability of the flow described by (21), if any, will first exist in these two cases.

For the first case, (21) for $\beta = 1$ has an explicit solution

$$\frac{c}{\Omega_1} = 1 - \frac{|\alpha-1|}{m(\alpha^2+1+2\alpha/\Delta)^{\frac{1}{2}}}\left\{1 \pm i\left[\frac{m(\alpha^2+1+2\alpha/\Delta)^{\frac{1}{2}}}{|\alpha-1|} - 1\right]^{\frac{1}{2}}\right\}. \tag{23}$$

This solution can also be obtained from (19). Since the expressions inside all the radicals are positive-definite, the azimuthal phase velocity is always complex, implying travelling instability, except for $\alpha = 1$, which corresponds to neutral stability. This conclusion obviously contradicts the findings in I (figures 2 and 3) that the flow was stable at least for the $m = 1$ mode when the sheet and the environment rotated at the same speed. As also supported by the previous discussion on instability characteristics, centrifugal instabilities due to the existence of negative density gradients in either the inner or the outer interface always occur except for homogeneous fluids.

The second special case we wish to examine is for $\alpha = 1$. Equation (21) has an exact solution

$$\frac{c}{\Omega_1} = 1 + \frac{\beta-1}{2}\left\{1 \pm \left[\frac{2\Delta+m^2(1-\Delta)}{m^2(1+\Delta)}\right]^{\frac{1}{2}} \pm i\left[\frac{2\Delta(m^2-1)}{m^2(1+\Delta)}\right]^{\frac{1}{2}}\right\}. \tag{24}$$

Obviously the azimuthal phase velocity is complex when $m \neq 1$ and the flow is always unstable, except for $m = 1$, which again is neutrally stable. Furthermore these unstable waves always travel when the sheet and the environment rotate in the same direction. This characteristic is predicted by the semi-ellipse theorem that no stationary disturbances can be unstable except when counter flows exist inside the flow domain. The finding in I (figures 2 and 3) showed that for homogeneous fluids all the stationary disturbances (with respect to the rotating environment) and some of the travelling disturbances were stable, at least for the $m = 1$ mode. The present analysis clearly shows that such a conclusion is incorrect.

It should be pointed out that (23) for $\beta = 1$ and (24) for $\alpha = 1$ are exact solutions to the secular relation (21) with no assumption on the complex phase-velocity and the middle-layer thickness. They clearly demonstrate that the dimensional relation between the two varies with the velocity and density distribution. Sufficient care should be taken if dimensional analyses are performed based on the complex phase velocity and the middle-layer thickness.

For flows with two interfaces, such as the one considered here, instabilities can take place at either the inner or the outer interface. Any stabilizing effect at one interface automatically implies destabilizing effect on the other interface as previously discussed. Stability of the system will only be neutral as in the cases shown by (23) and (24). In other words, stability, if any, will take place at the stability boundary.

3.3. *Two-region flow*

As the last special case to demonstrate the validity of the semicircle or semi-ellipse theorem discussed earlier, a two-region flow with constant properties in each region is being examined. By letting $\Omega_2 = \Omega_3$ and $\rho_2 = \rho_3$, the solution to the secular relation (18) is found to be

$$\frac{c_r}{\Omega_1} = \frac{(m-1)+\alpha\beta(m+1)}{m(\alpha+1)}, \tag{25a}$$

$$\frac{c_i}{\Omega_1} = \frac{\{\alpha[\beta(m+1)-(m-1)]^2-(\alpha+1)[\alpha\beta^2(m+1)-(m-1)]\}^{\frac{1}{2}}}{m(\alpha+1)}. \tag{25b}$$

Instabilities exist when the sum inside the square root is greater than zero. For unstable waves, the semicircle theorem, if valid, for the present profile reads

$$\left(\frac{c_r}{\Omega_1}-\frac{\beta+1}{2}\right)^2+\left(\frac{c_i}{\Omega_1}\right)^2 \leqslant \frac{(\beta-1)^2}{4}. \tag{26}$$

Substituting (25) into (26), we obtain

$$\frac{\beta\,(1-\alpha)}{m\,(1+\alpha)}+\frac{(\beta-1)^2}{4} \leqslant \frac{(\beta-1)^2}{4}. \tag{27}$$

The above inequality will be satisfied if

$$\beta(1-\alpha) \leqslant 0$$

or

$$\Omega_1\,\Omega_2(\rho_2-\rho_1) \geqslant 0. \tag{28}$$

It is clear that the semicircle described by (26) will no longer be the upper bound on the complex phase-velocity if (28) is violated. This is consistent with the restriction imposed on the derivation of the semi-ellipse theorem in (13), i.e. $ab(D\rho_0) \geqslant 0$. Of course, for the two-region flow under consideration, the semicircle theorem provides us with the best possible bound on the complex phase-velocity when (28) becomes an equality. The special case governed by (24) can also be shown to fall into this category.

4. Conclusions

The instability characteristics of a rotating sheet of fluid in a rotating environment have been re-investigated. Stability of the flow, if any, will have to be neutral. In other words, the stability domain only exists as the stability boundary. The stability boundary in this case corresponds to the $m = 1$ mode with no density inhomogeneity.

For non-axisymmetric instabilities of heterogeneous rotating flows, the semi-ellipse theorem holds with a restriction, i.e. when the product of the density gradient and the upper and lower bounds of the velocity is greater or equal to zero. It is demonstrated from the derivation of the semi-ellipse theorem that the unstable phase velocity will no longer be bounded by a semicircle if the restriction is violated. This argument is supported by an exact solution to the stability equation. For uniformly rotating flows, unstable waves occur when negative density gradients exist within the flow domain. In spite of the density variations such unstable waves always lie on a semicircle in the complex phase-velocity plane with diameter equal to the rotation speed.

REFERENCES

FUNG, Y. T. & KURZWEG, U. H. 1975 Stability of swirling flows with radius-dependent density. *J. Fluid Mech.* **72**, 243–255.

KOCHAR, G. T. & JAIN, R. K. 1979 Note on Howard's semicircle theorem. *J. Fluid Mech.* **91**, 489–491.

SAKURAI, T. 1976 Non-axisymmetric instability of a rotating sheet of gas in a rotating environment. *J. Fluid Mech.* **75**, 513–524.